AQA

AS
Further Mathematics

7366/1 Paper 1

Final Mark scheme

7366

June 2018

Version/Stage: v1.0

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
dM	mark is dependent on one or more M marks and is for method
R	mark is for reasoning
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
‘their’	Indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

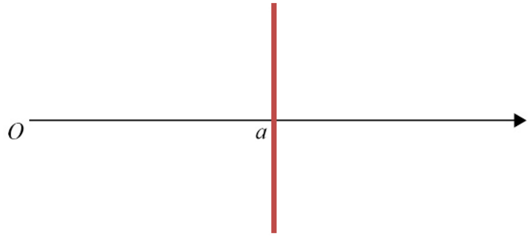
AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking instructions	AO	Mark	Typical solution
1	Circles correct answer	1.1b	B1	10
	Total		**1**	

Q	Marking instructions	AO	Mark	Typical solution
2	Circles correct answer	1.1a	B1	**AC**
	Total		**1**	

Q	Marking instructions	AO	Mark	Typical solution
3	Circles correct answer	1.2	B1	$\cos x$
	Total		**1**	

Q	Marking instructions	AO	Mark	Typical solution
4	Draws correct line. Condone a half-line.	1.1a	M1	
	Draws the full correct line and gives the correct intersection point on the initial line. Condone the angle omitted.	1.1b	A1	
	Total		**2**	

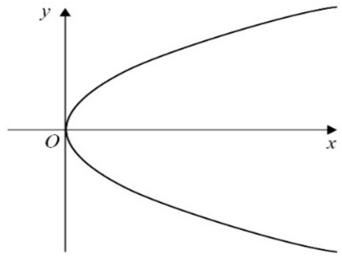
Q	Marking instructions	AO	Mark	Typical solution
5	Identifies correct values of sine **and** cosine.	1.1a	M1	$\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$
	Selects correct angle. Accept $\frac{2\pi}{3}$ or $-240°$ or $-\frac{4\pi}{3}$	1.1b	A1	$\theta = 120^{\circ}$
	Deduces the transformation giving a full description. FT their angle. Accept $\frac{2\pi}{3}$ or $-240°$ or $-\frac{4\pi}{3}$ Condone missing degree sign. Condone missing 'anticlockwise'. NMS scores 3/3	2.2a	A1F	Rotation about the z-axis through $120°$ anti-clockwise.
	Total		**3**	

Q	Marking instructions	AO	Mark	Typical solution
6a	Identifies the $\pm$ sign (or just the negative) as the error. May be seen in any of the last five lines. May be indicated within Matthew's solution or described in words. Ignore other 'errors' identified. Condone identifying any of the last five lines as containing the error. PI	2.3	B1	$\pm\sqrt{y^2+1} = e^x - y$
	Gives a correct reason, referring to either e^x (or e^y), or the operand of a logarithm, being positive. Do not award if more than one error identified.	2.4	E1	It is an error because $y - \sqrt{y^2+1} < 0$ and $e^x > 0$ so there is a contradiction.
6b	States the correct solution of the equation. Accept 10.0 [1787493] or $\frac{1}{2}(e^3 - e^{-3})$ ISW	1.1b	B1	$\operatorname{arsinh} x = 3$ $x = \sinh 3$
	Total		**3**	

Q	Marking instructions	AO	Mark	Typical solution
7	Obtains two equations in x and y. May be seen as a single vector equation. At least one equation must be correct. Accept a pair of letters other than x and y. Ignore any subsequent incorrect working.	1.1a	M1	$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 2x + 3y \\ x + 4y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $2x + 3y = x$ and $x + 4y = y$
	Obtains any two correct invariant points, with no incorrect points. Condone correct points given as position vectors. NMS: Correct answer scores 2/2. NMS: One correct invariant point and only one incorrect point scores SC1.	1.1b	A1	$x = -3y$ $\therefore$ two invariant points are $(0,0)$ and $(-3,1)$
	Total		**2**	

Q	Marking instructions	AO	Mark	Typical solution
8(a)	Correctly identifies the complex conjugate as a root.	1.1b	B1	2nd complex root is $2+3\text{i}$
	Forms one of the following equations (or their equivalents) $\alpha+\beta+2-3i=0$ or $\alpha\beta(2-3i)=\pm 52$ or $\alpha\beta+\beta(2-3i)+(2-3i)\beta=\pm m$ with an equation to find m. Or forms an equation to find m and then solves the cubic for their value of m.	1.1a	M1	sum of roots $=-\frac{b}{a}$ $\therefore\ \alpha+2+3\text{i}+2-3\text{i}=0$
	Finds correct third root	1.1b	A1	$\alpha+4=0$ $\alpha=-4$
				2nd complex root is $2+3\text{i}$ product of roots $=-\frac{d}{a}$ $\therefore\ \alpha(2+3\text{i})(2-3\text{i})=-52$ $\alpha(4-9i^2)=-52$ $\alpha=\dfrac{-52}{13}$ $\alpha=-4$

Q	Marking instructions	AO	Mark	Typical solution
8(b)	Forms a correct equation to find m. May be seen in part (a).	1.1a	M1	$\therefore\ (-4)^3 + m(-4) + 52 = 0$
	Finds correct value of m.	1.1b	A1	$-64 - 4m + 52 = 0$ $-12 = 4m$ $m = -3$
				$\sum \alpha\beta = \frac{c}{a}$ $\therefore\ (2-3i)(2+3i) + (-4)(2-3i)$ $+(-4)(2+3i) = m$ $4 + 9 - 8 + 12i - 8 - 12i = m$ $m = -3$
	Total		**5**	

Q	Marking instructions	AO	Mark	Typical solution
9(a)	Sketches correct parabola	1.2	B1	
9(b)(i)	Obtains $\pi \int 4x\,dx$ Limits not required for this mark. Condone missing dx	3.3	M1	volume $= \pi \int_0^d y^2\,dx$ $\therefore\ 1000 = \pi \int_0^d 4x\ dx$
	Obtains $2x^2$ and uses limits of d and 0 (oe).	1.1b	B1	$\frac{1000}{\pi} = \left[\frac{4x^2}{2}\right]_0^d$ $\frac{1000}{\pi} = 2d^2 - 0$
	Forms an equation of the form $kd^2 = \text{volume}$ (oe)	3.4	M1	$2\pi d^2 = 1000$
	Correct depth to nearest millimetre. Condone 126 or 12.6 without units. NMS: 126 or 12.6 scores 4/4. Using 1000000 mm^3 leads to a correct answer of 399 mm for 4/4.	3.2a	A1	$d = \sqrt{\frac{500}{\pi}}$ depth = 12.6 cm
9(b)(ii)	States appropriate assumption	3.5b	B1	The thickness of the plastic is negligible
	Total		**6**	

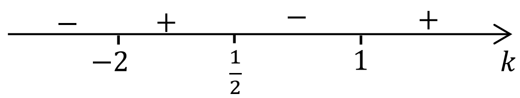
Q	Marking instructions	AO	Mark	Typical solution
10(a)	Demonstrates the rule is correct for $n = 1$	1.1b	B1	$\sum_{r=1}^{1} r^3 = 1^3 = 1$ and $\frac{1}{4} \times 1^2 \times 2^2 = 1$ $\therefore$ it is true for $n = 1$
	States the rule is true for $n = k$ **and** adds $(k+1)^3$ to $\frac{1}{4}k^2(k+1)^2$	2.4	M1	Assume it is true for $n = k$ $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$ $\sum_{r=1}^{k} r^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$
	Obtains $\frac{1}{4}(k+1)^2(k+2)^2$ from $\frac{1}{4}k^2(k+1)^2 + (k+1)^3$	2.2a	A1	$\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right)$ $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$ $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$
	Completes a rigorous argument and explains how their argument proves the required result, This mark is only available if all previous marks have been awarded.	2.1	R1	$\therefore$ it is also true for $n = k + 1$ True for $n = 1$, and true for $n = k \Rightarrow$ true for $n = k + 1$, then by induction it is true for all integers $n \geq 1$ **AG**

Q	Marking instructions	AO	Mark	Typical solution
10(b)	Expresses LHS as summations of r^3 and r Ignore limits of the sums in this part only.	1.1b	B1	$\sum_{r=1}^{2n} r(r-1)(r+1) = \sum_{r=1}^{2n}(r^3 - r)$ $= \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{2n} r$
	Expresses LHS in terms of n, using part (a) and $\sum r = \frac{1}{2}n(n+1)$	1.1a	M1	$= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{2}2n(2n+1)$
	Takes out $n(2n+1)$ as a factor or obtains $4n^4 + 4n^3 - n^2 - n$ Allow one slip in second bracket or one incorrect term in the expansion.	1.1a	M1	$= n^2(2n+1)^2 - n(2n+1)$ $= n(2n+1)(n(2n+1) - 1)$
	Completes fully correct proof to reach the required result. This mark is only available if all previous marks have been awarded. Note: $n(n+1)(2n-1)(2n+1) = 4n^4 + 4n^3 - n^2 - n$	2.1	R1	$= n(2n+1)(2n^2 + n - 1)$ $= n(2n+1)(2n-1)(n+1)$ $= n(n+1)(2n-1)(2n+1)$ **AG**
	Total		**8**	

Q	Marking instructions	AO	Mark	Typical solution
11	States integral(s) of the cubic with limits that include 1 and 4	1.1a	M1	$\text{mean} = k = \frac{1}{4-1}\int_1^4 (x^3 - 7x^2 + 11x + 6)\ dx$
	Integrates the function and substitutes correct limits. Condone one incorrect term. Note: $\int_1^4 (x^3 - 7x^2 + 11x + 6)\ dx = \frac{69}{4} \Rightarrow \text{M1M1}$	1.1a	M1	$\therefore\ k = \frac{1}{3}\left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{11x^2}{2} + 6x\right]_1^4$ $= \frac{1}{3}\left(\frac{4^4}{4} - \frac{7\times 4^3}{3} + \frac{11\times 4^2}{2} + 6\times 4\right) - \frac{1}{3}\left(\frac{1^4}{4} - \frac{7\times 1^3}{3} + \frac{11\times 1^2}{2} + 6\times 1\right)$
	Obtains the correct value of k. Do not apply ISW. NMS can score 3/3.	1.1b	A1	$= \frac{1}{3}\times\frac{80}{3} - \frac{1}{3}\times\frac{113}{12}$ $= 5.75$
	Total		**3**	

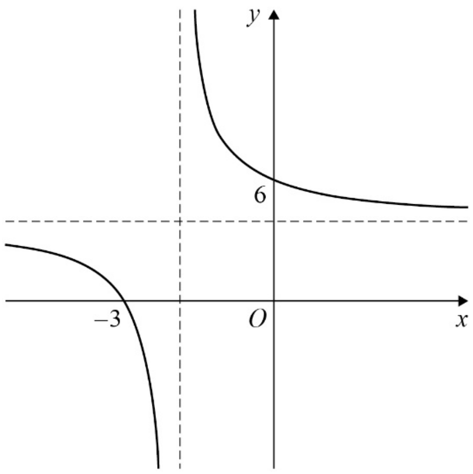
		$\int_1^4 (x^3 - 7x^2 + 11x + 6 - k)\ dx = 0$ $\left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{11x^2}{2} + 6x - kx\right]_1^4 = 0$ $\left(\frac{4^4}{4} - \frac{7 \times 4^3}{3} + \frac{11 \times 4^2}{2} + 6 \times 4 - k \times 4\right) - \left(\frac{1}{4} - \frac{7}{3} + \frac{11}{2} + 6 - k\right) = 0$ $64 - \frac{448}{3} + 88 + 24 - 4k - \frac{113}{12} + k = 0$ $\frac{69}{4} = 3k$ $k = 5.75$

		Area B = Area C $\therefore \int_1^p (x^3 - 7x^2 + 11x + 6 - k)\ dx = \int_p^4 (k - x^3 + 7x^2 - 11x - 6)\ dx$ $\left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{11x^2}{2} + 6x - kx\right]_1^p = \left[kx - \frac{x^4}{4} + \frac{7x^3}{3} - \frac{11x^2}{2} - 6x\right]_p^4$ $\left(\frac{p^4}{4} - \frac{7p^3}{3} + \frac{11p^2}{2} + 6p - kp\right) - \left(\frac{1}{4} - \frac{7}{3} + \frac{11}{2} + 6 - k\right)$ $= \left(4k - 64 + \frac{7 \times 64}{3} - \frac{11 \times 16}{2} - 24\right) - \left(kp - \frac{p^4}{4} + \frac{7p^3}{3} - \frac{11p^2}{2} - 6p\right)$ $-\frac{1}{4} + \frac{7}{3} - \frac{11}{2} - 6 + k = 4k - 64 + \frac{448}{3} - 88 - 24$ $\frac{69}{4} = 3k$ $k = 5.75$

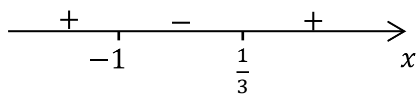
Q	Marking instructions	AO	Mark	Typical solution
12(a)	Substitutes $k = 1$ and correctly calculates the determinant, and concludes that the matrix is singular.	2.2a	B1	determinant $= 4 \times 1 - 2 \times 2 = 0$ $\therefore$ the matrix is singular **AG**
12(b)	Finds determinant in terms of k. Allow one error.	1.1a	M1	determinant $= k(5-k) - 2(k^3+1)$
	Obtains a correct inequality in k.	1.1b	A1	$5k - k^2 - 2k^3 - 2 < 0$
	Obtains three correct critical values.	1.1b	A1	$2k^3 + k^2 - 5k + 2 > 0$ $(k-1)(2k^2+3k-2) > 0$ $(k-1)(2k-1)(k+2) > 0$ 
	Deduces one correct region. FT their three real distinct critical values if given as $a < k < b$, $k > c$ (o.e.) where $a < b < c$	2.2a	A1F	
	Deduces the other correct region. FT their three real distinct critical values if given as $a < k < b$, $k > c$ (o.e.) where $a < b < c$ Condone the use of ‘and’.	2.2a	A1F	$-2 < k < \frac{1}{2}$, $k > 1$
	Total		**6**	

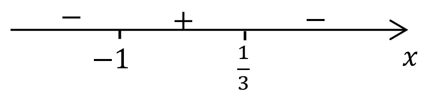
<table>
<tr><th>Q</th><th>Marking instructions</th><th>AO</th><th>Mark</th><th>Typical solution</th></tr>
<tr><td rowspan="2">13(a)</td><td>Writes any rational function with a horizontal asymptote of $y = 2$ or one vertical asymptote of $x = -1$,
e.g. $y = \frac{ax+b}{x+1}$ or $y = \frac{2x+b}{x+c}$
or $y = \frac{ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots}{dx^n+ex^{n-1}+fx^{n-2}+\ldots\ldots}$ where $\frac{a}{d} = 2$
Accept any correct rearrangement of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a function as described above.</td><td>3.1a</td><td>M1</td><td rowspan="2">$y = \frac{2x+m}{x+1}$

but $x = -3$ when $y = 0$
$\therefore\ 0 = \frac{2 \times -3 + c}{-3+1}$
$-6 + c = 0$
$c = 6$
$\therefore\ y = \frac{2x+6}{x+1}$</td></tr>
<tr><td>Obtains a fully correct answer.</td><td>1.1b</td><td>A1</td></tr>
<tr><td></td><td colspan="3"></td><td>$(x+1)(y-2) = n$
but $x = -3$ when $y = 0$
$\therefore\ (-3+1)(0-2) = n$
$4 = n$
$\therefore\ (x+1)(y-2) = 4$
$y - 2 = \frac{4}{x+1}$
$y = 2 + \frac{4}{x+1}$</td></tr>
</table>

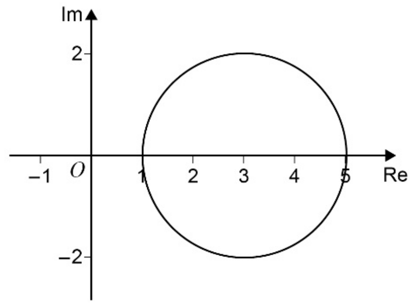
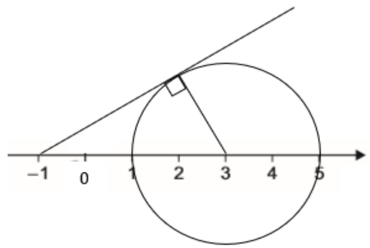
Q	Marking instructions	AO	Mark	Typical solution
13(b)	Sketches any rectangular hyperbola, or rational function, tending to the correct vertical and horizontal asymptotes included or implied.	1.1a	M1	
	Sketches a correct graph, including the asymptotes. Accept the graph of their function if M1A1 scored in part (a). Accept un-ruled asymptotes – mark intention.	1.1b	A1	
	Indicates correct axis-intercepts. Follow through their equation if their y-intercept matches their graph.	1.1b	A1F	

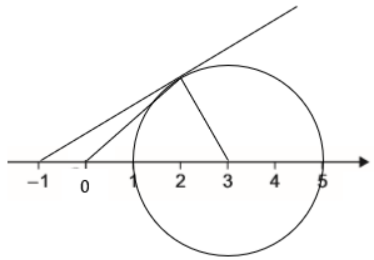
Q	Marking instructions	AO	Mark	Typical solution
13(c)	Forms an equation or inequality with $y = 5$ and their rational function.	1.1a	M1	$5 = \dfrac{2x+6}{x+1}$
	Obtains correct x-intercept with $y = 5$ Follow through their rational function from part (a)	1.1b	A1F	$5(x+1) = 2x+6$ $5x+5 = 2x+6$ $3x = 1$ $x = \dfrac{1}{3}$
	Deduces one correct region $x \geq \frac{1}{3}$ **or** $x < -1$ Condone $x \leq -1$ for this mark. Follow through their $\frac{1}{3}$ if greater than –1	2.2a	A1F	$x < -1\ ,\ x \geq \dfrac{1}{3}$
	Deduces correct regions. Accept correct regions for their function if M1A1 scored in part (a).	2.2a	A1	

Q	Marking instructions	AO	Mark	Typical solution
				$\dfrac{2x+6}{x+1} \le 5$ $(2x+6)(x+1) \le 5(x+1)^2$ $0 \le (x+1)\big(5(x+1)-(2x+6)\big)$ $0 \le (x+1)(3x-1)$  $x < -1\ ,\ \ x \ge \dfrac{1}{3}$

Q	Marking instructions	AO	Mark	Typical solution
				$\frac{2x+6}{x+1} \leq 5$ $\frac{2x+6}{x+1} - \frac{5(x+1)}{x+1} \leq 0$ $\frac{-3x+1}{x+1} \leq 0$  $x < -1\ ,\ \ x \geq \frac{1}{3}$

Q	Marking instructions	AO	Mark	Typical solution
				For $x > -1$: $2x + 6 \le 5(x + 1)$ $1 \le 3x$ $x \ge \frac{1}{3}$ For $x < -1$: $2x + 6 \ge 5(x + 1)$ $1 \ge 3x$ $x < -1 \ , \ x \ge \dfrac{1}{3}$
	Total		**9**	

Q	Marking instructions	AO	Mark	Typical solution
14(a)	Draws a circle with centre $(3, 0)$ and radius 2. Accept freehand circle. Ignore any straight lines drawn on the diagram.	1.1b	B1	
14(b)(i)	Uses fully correct method for $sin\alpha$ or $cos\alpha$ or $tan\alpha$ $\sin\alpha = \frac{2}{4}$ $\cos\alpha = \frac{\sqrt{4^2-2^2}}{4}$ $\tan\alpha = \frac{2}{\sqrt{4^2-2^2}}$	3.1a	M1	 $\sin\alpha = \frac{2}{4}$
	Obtains correct value for α Accept 0.52(35987756) Condone 30°	1.1b	A1	$\alpha = \sin^{-1}\left(\frac{2}{4}\right)$ $\alpha = \frac{\pi}{6}$

14(b)(ii)	Forms an equation in r using cosine rule or equivalent. Follow through their α. Or forms a correct equation in x and y.	3.1a	M1	 $r^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos\frac{\pi}{3}$ $r = \sqrt{7}$
	Forms an equation in θ using sine rule or equivalent. Follow through their α. Or forms a second correct equation in x and y.	1.1a	M1	$\frac{sin\theta}{2} = \frac{sin\frac{\pi}{3}}{\sqrt{7}}$
	Obtains correct value for r **or** θ. Or obtains correct values for x **and** y.	1.1b	A1	$sin\theta = 2 \times \frac{\sqrt{3}}{2} \div \sqrt{7} = \frac{\sqrt{21}}{7}$ $\theta = 0.71$
	Expresses w in the required form. Accept $2.6\,[45751311]$ or $\sqrt{7}$ for r Accept 0.71[37243789] or $\sin^{-1}\left(\frac{\sqrt{21}}{7}\right)$ or $\sin^{-1}\left(\sqrt{\frac{3}{7}}\right)$ for θ	1.1b	A1	$w = 2.6(cos0.71 + isin0.71)$
	Total		**7**	

Q	Marking instructions	AO	Mark	Typical solution
15(a)	Shows the result is true with at least one intermediate step.	1.1b	B1	$\frac{1}{r+2}-\frac{1}{r+3}=\frac{r+3-(r+2)}{(r+2)(r+3)}$ $=\frac{1}{(r+2)(r+3)}$
15(b)	Writes at least three corresponding terms of $\frac{1}{r+2}$ and $\frac{1}{r+3}$ Must include the 1st and nth terms and at least the 2nd term or the $(n-1)^{\text{th}}$ term.	1.1a	M1	$\sum_{r=1}^{n}\left(\frac{1}{(r+2)(r+3)}\right)=\sum_{r=1}^{n}\left(\frac{1}{r+2}-\frac{1}{r+3}\right)$ $=\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots\cdots$ $+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$
	Correctly uses the method of differences to reduce the sum to two terms.	1.1b	A1	$=\frac{1}{3}-\frac{1}{n+3}$
	Completes fully correct proof to reach the required result. This mark is only available if all previous marks have been awarded.	2.1	R1	$=\frac{n+3-3}{3(n+3)}$ $=\frac{n}{3(n+3)}$ **AG**
	Total		**4**	

Q	Marking instructions	AO	Mark	Typical solution
16	Uses factorisation or pre-multiplication to isolate $\boldsymbol{A}$	3.1a	M1	$\boldsymbol{AB} - 2\boldsymbol{A} = \boldsymbol{I}$ $\boldsymbol{A}(\boldsymbol{B} - 2\boldsymbol{I}) = \boldsymbol{I}$
	Deduces $\boldsymbol{A}$ in terms of $\boldsymbol{B}$ and $\boldsymbol{I}$. Could be implied by sight of $\begin{bmatrix} 1 & -2 \\ -4 & 6 \end{bmatrix}$ with attempt to invert.	2.2a	A1	$\boldsymbol{A} = (\boldsymbol{B} - 2\boldsymbol{I})^{-1}$
	Obtains correct matrix $\boldsymbol{A}$.	1.1b	A1	$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -4 & 6 \end{bmatrix}^{-1}$ $\boldsymbol{A} = \frac{1}{-2}\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix}$ $\boldsymbol{A} = \begin{bmatrix} -3 & -1 \\ -2 & -0.5 \end{bmatrix}$

ALT 16	Sets up four equations with at least three correct.	3.1a	M1	Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $3a - 4b = 1 + 2a$ and $3c - 4d = 0 + 2c$ and $-2a + 8b = 0 + 2b$ and $-2c + 8d = 1 + 2d$
	Deduces at least two correct elements of $\boldsymbol{A}$. Note: Two correct elements from just two correct equations can score M1A1.	2.2a	A1	$a = 4b + 1$ and $c = 4d$ $6b = 2a$ and $6d = 2c + 1$ $\therefore$ $3b = 4b + 1$ and $6d = 2(4d) + 1$ $-1 = b$ and $-1 = 2d \Rightarrow d = -\frac{1}{2}$ $\therefore$ $a = 4 \times -1 + 1$ and $c = 4 \times -\frac{1}{2}$ $a = -3$ and $c = -2$
	Obtains correct matrix $\boldsymbol{A}$	1.1b	A1	$\therefore$ $\boldsymbol{A} = \begin{bmatrix} -3 & -1 \\ -2 & -0.5 \end{bmatrix}$
	Total		**3**	

Q	Marking instructions	AO	Mark	Typical solution
17	Recalls and uses $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ **and** $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ PI	1.2	B1	$\frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \times \left(\frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) + \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)\right) = 1$
	Forms equation and rearranges to obtain exactly one exponential term	3.1a	M1	$\frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \times \left(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{\theta}\right) = 1$ $e^{2\theta} - e^{0} = 2$
	Takes logarithms of an equation of the form $e^{2\theta} = k$ where $k > 0$	1.1a	M1	$e^{2\theta} = 3$ $2\theta = \ln 3$
	Obtains correct answer in required form	1.1b	A1	$\theta = \frac{1}{2}\ln 3$

ALT 17	Use of $cosh^2\theta - sinh^2\theta = 1$ PI	3.1a	M1	$sinh^2\theta + sinh\theta cosh\theta = cosh^2\theta - sinh^2\theta$
	Recalls and uses $tanh\theta = \frac{sinh\theta}{cosh\theta}$	1.2	B1	$\frac{sinh^2\theta}{cosh^2\theta} + \frac{sinh\theta cosh\theta}{cosh^2\theta} = \frac{cosh^2\theta}{cosh^2\theta} - \frac{sinh^2\theta}{cosh^2\theta}$ $tanh^2\theta + tanh\theta = 1 - tanh^2\theta$
	Solves a three-term quadratic in $tanh\theta$ (oe)	1.1a	M1	$2tanh^2\theta + tanh\theta - 1 = 0$ $(2tanh\theta - 1)(tanh\theta + 1) = 0$ $tanh\theta = \frac{1}{2}$ or $tanh\theta = -1$
	Obtains the correct answer. ISW Condone lack of reference to $tanh\theta \neq -1$	1.1b	A1	but $tanh\theta \neq -1 \therefore tanh\theta = \frac{1}{2}$ only $\theta = \text{artanh}\left(\frac{1}{2}\right)$
	Total		**4**	

Q	Marking instructions	AO	Mark	Typical solution
18	Writes the expression in terms of $\sum \alpha$ and $\sum \alpha\beta$ Award for correct expansion followed by use of $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$	3.1a	M1	$(\alpha-\beta)^2 + (\gamma-\alpha)^2 + (\beta-\gamma)^2$ $= \alpha^2 - 2\alpha\beta + \beta^2 + \gamma^2 - 2\gamma\alpha + \alpha^2 + \beta^2 - 2\beta\gamma + \gamma^2$ $= 2\alpha^2 + 2\beta^2 + 2\gamma^2 - 2\alpha\beta - 2\gamma\alpha - 2\beta\gamma$ $= 2\sum \alpha^2 - 2\sum \alpha\beta$ $= 2((\sum \alpha)^2 - 2\sum \alpha\beta) - 2\sum \alpha\beta$
	Substitutes $\pm m$ for $\sum \alpha$ **and** $\pm n$ for $\sum \alpha\beta$	1.1a	M1	$= 2(\sum \alpha)^2 - 6\sum \alpha\beta$ $= 2(-m)^2 - 6 \times n = 2m^2 - 6n$
	Gives a reason for expression ≥ 0 Condone lack of reference to roots being real.	2.4	E1	But as α, β and γ are real then each of $(\alpha-\beta)^2$, $(\gamma-\alpha)^2$ and $(\beta-\gamma)^2$ must be non-negative. $\therefore (\alpha-\beta)^2 + (\gamma-\alpha)^2 + (\beta-\gamma)^2 \geq 0$
	Completes fully correct proof to reach the required result. This mark is only available if all previous marks have been awarded. Lose this mark for sight of $\sum \alpha = m$	2.1	R1	$\therefore 2m^2 - 6n \geq 0$ $2m^2 \geq 6n$ $m^2 \geq 3n$ **AG**
	Total		**4**	

Q	Marking instructions	AO	Mark	Typical solution
19(a)	Finds a direction vector for the second wire. Condone one error.	3.4	M1	Direction vector for 2nd wire = $\begin{pmatrix}10\\0\\20\end{pmatrix}-\begin{pmatrix}-10\\100\\-5\end{pmatrix}$
	Writes, in terms of a parameter, the position vector (or coordinates) of one point on each of the two lines. Condone use of same parameter.	3.1a	M1	$\boldsymbol{r_1}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\100\\-20\end{pmatrix}$ and $\boldsymbol{r_2}=\begin{pmatrix}10\\0\\20\end{pmatrix}+\mu\begin{pmatrix}20\\-100\\25\end{pmatrix}$
	Obtains, in terms of two parameters, a correct vector between the two lines.	1.1b	A1	$\boldsymbol{r_2}-\boldsymbol{r_1}=\begin{pmatrix}10+20\mu\\-100\mu-100\lambda\\20+25\mu+20\lambda\end{pmatrix}$
	Sets up two scalar products for their $\boldsymbol{r_2}-\boldsymbol{r_1}$ and their valid direction vectors.	1.1a	M1	$\begin{pmatrix}10+20\mu\\-100\mu-100\lambda\\20+25\mu+20\lambda\end{pmatrix}.\begin{pmatrix}0\\100\\-20\end{pmatrix}=0$ and $\begin{pmatrix}10+20\mu\\-100\mu-100\lambda\\20+25\mu+20\lambda\end{pmatrix}.\begin{pmatrix}20\\-100\\25\end{pmatrix}=0$
	Obtains correct parameter values.	1.1b	A1	$-10000\mu-10000\lambda-400-500\mu-400\lambda=0$ and $200+400\mu+10000\mu+10000\lambda+500+625\mu+500\lambda=0$ $11025\mu+10500\lambda+700=0$ and $11025\mu+10920\lambda+420=0$ $\lambda=\frac{2}{3}$ and $\mu=-\frac{44}{63}$
	Uses full method for required distance	1.1b	M1	$\sqrt{\left(10+20\left(-\frac{44}{63}\right)\right)^2+\left(-100\left(-\frac{44}{63}\right)-100\left(\frac{2}{3}\right)\right)^2+\left(20+25\left(-\frac{44}{63}\right)+20\left(\frac{2}{3}\right)\right)^2}$
	Obtains correct distance to 2, 3 or 4 significant figures with correct units. Accept 1 significant figure if full method shown.	3.2a	A1	= 16.7 metres

Q	Marking instructions	AO	Mark	Typical solution
ALT 19(a)	Finds a direction vector for the second wire.	3.4	M1	Direction vector for 2nd wire = $\begin{pmatrix}10\\0\\20\end{pmatrix}-\begin{pmatrix}-10\\100\\-5\end{pmatrix}$
	Forms two equations for a perpendicular vector	3.1a	M1	Let $\begin{pmatrix}x\\y\\z\end{pmatrix}$ be a vector perpendicular to both wires. $\therefore \begin{pmatrix}0\\100\\-20\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$ **and** $\begin{pmatrix}20\\-100\\25\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$ $\Rightarrow 100y-20z=0$ **and** $20x-100y+25z=0$
	Obtains two correct equations for a perpendicular vector	1.1b	A1	$\Rightarrow z=5y$ **and** $x=-1.25y$
	Obtains a correct normal vector	1.1b	A1	$\therefore$ perpendicular vector is $\begin{pmatrix}-1.25y\\y\\5y\end{pmatrix}$
	Finds the unit normal vector	1.1a	M1	$\Rightarrow$ unit perpendicular vector is $\begin{pmatrix}-1.25\\1\\5\end{pmatrix}\div\sqrt{(-1.25)^2+1^2+5^2}$
	Uses full method for required distance	1.1b	M1	a vector from 1st line to 2nd line is $\begin{pmatrix}10\\0\\20\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}10\\0\\20\end{pmatrix}$ $\therefore$ distance between lines is $\begin{pmatrix}10\\0\\20\end{pmatrix}.\begin{pmatrix}-1.25\\1\\5\end{pmatrix}\div\frac{21}{4}$
	Obtains correct distance to 2, 3 or 4 significant figures with correct units. Accept 1 significant figure if full method shown.	3.2a	A1	= 16.7 metres

19(b)	Suggests an **improvement** to the model. Do not condone criticisms without refinements.	3.5c	B1	Model the wires as curves
	Total		**8**	
	TOTAL		**80**	